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Extremal Properties of Ultraspherical Polynomials

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For the polynomials $P_l(x)=a_{l0}+a_{l1}x+\cdots+a_{ll}x^l$ (of degree l) we consider the problem of maximizing a weighted product of the absolute values of the highest coefficients $\Pi_{l=1}^n |a_{ll}|^{\beta_l}$ among all polynomials $P_1, ..., P_n$ for which the weighted sum of squares $\sum_{l=1}^n \beta_l P_l^2(x)$ is bounded by 1 on the interval $[-1, 1]$. By an application of a duality result the solutions (depending on the weights $\beta_l \geqslant 0$) of these problems are determined. The "optimal" polynomials are the orthonormal polynomials with respect to a probability measure minimizing a weighted product of determinants of Hankel matrices (the solution of the dual problem). For a special class of weights $\beta_1, ..., \beta_n$ the optimal polynomials can be represented in terms of ultraspherical polynomials. Thus some new extremal properties are obtained for these polynomials which generalize the well known fact that among all polynomials P_n of degree n with $|P_n(x)| \leqslant 1$ (on $[-1, 1]$) the maximum of the highest coefficient is obtained for the Chebyshev polynomial of the first kind. The results are illustrated in several examples. © 1994 Academic Press, Inc.

1. INTRODUCTION

Consider the well known Chebyshev polynomials

$$T_n(x)=\cos(n \text{ arc} \cos x)$$

and

$$U_n(x)=\frac{\sin((n+1) \text{ arc} \cos x)}{\sin(\text{arc} \cos x)}$$

($x \in [-1, 1]$) of the first and second kind which are the orthogonal polynomials (with leading coefficients 2^{n-1} and 2^n) with respect to the measures $(1-x^2)^{-1/2}\,dx$ and $(1-x^2)^{1/2}\,dx$, respectively (see, for example, Szegö [22, p. 60]). One of the most beatiful features of these polynomials is that they are the solutions of many extremal problems involving polynomials (see, e.g., Rivlin [12, p. 92] or Natanson [9, p. 50]). In this paper we

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are interested in extremal problems involving the highest coefficients of polynomials which satisfy a restriction in the sup-norm. More precisely, it is well known (see, e.g., Karlin and Studden [7, p. 310], [12, p. 93], or [9, p. 50]) that among all polynomials $P_n(x) = a_{n0} + a_{n1}x + \cdots + a_{nn}x^n$ satisfying $\max_{x\in[-1,1]} |P_n(x)| \leqslant 1$ the maximum of the highest coefficient (i.e., the coefficient of x^n) is obtained for the Chebyshev polynomial of the first kind (here the double index for the coefficients a_{nl} of the polynomial $P_n(x)$ is used to be consistent with later notation). Similarly, $U_n(x)$ maximizes the absolute value of the highest coefficient among all polynomials P_n satisfying $\max_{x\in[-1,1]} \sqrt{1-x^2}\, |P_n(x)| \leqslant 1$.

It is the purpose of this paper to consider some generalizations of the above extremal problems. To this end let $\mathbb{P}_n$ be the set of polynomials up to degree $n \in \mathbb{N}$ and let for $l = 0, ..., n$ $P_l(x) = a_{l0} + a_{l1}x + \cdots + a_{ll}x^l$ denote an arbitrary element of $\mathbb{P}_l$. For $n \in \mathbb{N}$ we define the set

$$\mathscr{P}_n := \left\{ (P_1, ..., P_n) \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_n \,\middle|\, \max_{x\in[-1,1]} \sum_{l=1}^{n} \beta_l P_l^2(x) \leqslant 1 \right\}, \tag{1.1}$$

where $\beta_1, ..., \beta_n$ are nonnegative (given) weights with sum 1. For $P_l(x) = a_{l0} + a_{l1}x + \cdots + a_{ll}x^l \in \mathbb{P}_l$ let $m_l(P_l) = a_{ll}$ denote the coefficient of x^l in the polynomial P_l. We are asking for the solution of the problem

$$\sup \left\{ \prod_{l=1}^{n} |m_l(P_l)|^{\beta_l} \,\middle|\, (P_1, ..., P_n) \in \mathscr{P}_n \right\} \tag{P}$$

(if $\beta_l = 0$ we define $|m_l(P_l)|^{\beta_l} = 1$ for all $P_l \in \mathbb{P}_l$ and do not consider the polynomials of degree l in the set $\mathscr{P}_n$). In the same way we define

$$\mathscr{Q}_n := \left\{ (Q_0, ..., Q_n) \in \mathbb{P}_0 \times \cdots \times \mathbb{P}_n \,\middle|\, \max_{x\in[-1,1]} (1-x^2) \sum_{l=0}^{n} \beta_l Q_l^2(x) \leqslant 1 \right\} \tag{1.2}$$

and we consider the extremal problem

$$\sup \left\{ \prod_{l=0}^{n} |m_l(Q_l)|^{\beta_l} \,\middle|\, (Q_0, ..., Q_n) \in \mathscr{Q}_n \right\}. \tag{Q}$$

Extremal problems of the form (P) and (Q) appear in the geometric solution of model robust design problems in linear regression models. Here the coefficients of the polynomials $P_1, ..., P_n$ in the set $\mathscr{P}_n$ define a covering halfspace to a convex subset of $\mathbb{R}^{n(n+1)/2}$ and a function depending on these coefficients has to be maximized over the set of all these covering halfspaces (see Dette [3] for more details).

Obviously, (P) and (Q) reduce to the classical extremal problems for the Chebyshev polynomials of the first and second kind in the case $\beta_n = 1$ and $\beta_l = 0$ if $l \leqslant n-1$. While the solution of these "original" problems is more or less elementary (see [9, p. 50]) it is more complicated to determine the maximizing "set" of polynomials for the generalizations (P) and (Q). In Section 2 we will identify dual problems (P*) and (Q*) corresponding to (P) and (Q) as minimization problems for determinants of Hankel matrices in the set of all probability measures on the interval $[-1, 1]$. A strong duality theorem is proved and the solution of the dual problems (P*) and (Q*) (i.e., the minimizing probability measure ξ^*) is determined in terms of canonical moments which were used by Studden [19] for the solution of a generalized problem of Chebyshev and by Lau and Studden [8] for the solution of Fejér's problem. The maximizing polynomials in the problems (P) and (Q) are then the orthonormal polynomials with respect to the probability measure solving the corresponding dual problem. In Sections 3 and 4 we consider "special" weight sequences (e.g., the uniform weights $\beta_1 = \cdots = \beta_n = 1/n$) for which the solutions of the problems (P*) and (Q*) become more transparent. For these sequences the maximizing polynomials are proportional to weighted sums of ultraspherical polynomials and we can represent the solutions of (P) and (Q) in terms of these classical polynomials. Some generalizations of the results are given in Section 5. There we consider weights $\beta_1, ..., \beta_n$ where most of the β_l are vanishing. Especially we show that among all polynomials $P_k(x)$ and $P_{2k}(x)$ of degree k and $2k$ satisfying ($\beta_1 \in (0, 1]$)

$$(1-\beta_1)\, P_k^2(x) + \beta_1 P_{2k}^2(x) \leqslant 1 \qquad \text{for all} \quad x \in [-1, 1]$$

the maximum value of the weighted product of the absolute values of the highest coefficients $|m_k(P_k)|^{1-\beta_1}\, |m_{2k}(P_{2k})|^{\beta_1}$ is obtained for a linear and quadratic polynomial in the kth Chebyshev polynomial of the first kind $T_k(x)$. Some of the more complicated proofs of the results are given in Appendixes A and B.

2. The Dual Problem and Its Solution

Throughout this paper $f_l(x) = (1, x, ..., x^l)' \in \mathbb{R}^{l+1}$ will denote the vector of monomials up to the order l ($l = 0, ..., n$) and ξ stands for a probability measure on the interval $[-1, 1]$ (or on its Borel field) with moments $c_i = \int_{-1}^{1} x^i \, d\xi(x)$ ($i \geqslant 0$). We introduce the well known "Hankel" matrices (see, e.g., Karlin and Shapely [6, p. 56] or [7, p. 106])

$$\underline{M}_{2l}(\xi)=\int_{-1}^{1} f_l(x)\, f_l'(x)\, d\xi(x)$$

$$=\begin{bmatrix} c_0 & c_1 & \cdots & c_l \\ c_1 & c_2 & \cdots & c_{l+1} \\ \vdots & \vdots & & \vdots \\ c_l & c_{l+1} & \cdots & c_{2l} \end{bmatrix} \tag{2.1}$$

$$\bar{M}_{2l+2}(\xi)=\int_{-1}^{1} f_l(x)\, f_l'(x)(1-x^2)\, d\xi(x)$$

$$=\begin{bmatrix} c_0-c_2 & c_1-c_3 & \cdots & c_l-c_{l+2} \\ c_1-c_3 & c_2-c_4 & \cdots & c_{l+1}-c_{l+3} \\ \vdots & \vdots & & \vdots \\ c_l-c_{l+2} & c_{l+1}-c_{l+3} & \cdots & c_{2l}-c_{2l+2} \end{bmatrix} \tag{2.2}$$

and their corresponding determinants $\underline{\Delta}_{2l}(\xi)=\det(\underline{M}_{2l}(\xi))$, $\bar{\Delta}_{2l+2}(\xi)=\det(\bar{M}_{2l+2}(\xi))$. For nonnegative weights β_l with sum 1 and $\beta_n>0$ we consider the minimization problems

$$(\mathrm{P}^*) \qquad \inf\left\{\prod_{l=1}^{n}\left(\frac{\underline{\Delta}_{2l-2}(\xi)}{\underline{\Delta}_{2l}(\xi)}\right)^{\beta_l} \,\middle|\, \xi\in\underline{\Xi}\right\},$$

$$(\mathrm{Q}^*) \qquad \inf\left\{\prod_{l=0}^{n}\left(\frac{\bar{\Delta}_{2l}(\xi)}{\bar{\Delta}_{2l+2}(\xi)}\right)^{\beta_l} \,\middle|\, \xi\in\bar{\Xi}\right\}$$

($\underline{\Delta}_0=\bar{\Delta}_0=1$), where $\underline{\Xi}$ (and $\bar{\Xi}$) denote the set of all probability measures on the interval $[-1, 1]$ such that the matrices $\underline{M}_{2l}(\xi)$ (and $\bar{M}_{2l+2}(\xi)$) are nonsingular ($l=0, ..., n$). The essential step for the solution of the "primal" problems (P) and (Q) is the following duality theorem whose proof is complicated and therefore deferred to the Appendix.

THEOREM 2.1 (Duality). *The problems* (P*) *and* (Q*) *are the dual problems of* (P) *and* (Q). *More precisely, we have*

$$\max\left\{\prod_{l=1}^{n} |m_l(P_l)|^{2\beta_l} \,|\, (P_1, ..., P_n)\in\mathscr{P}_n\right\}$$

$$=\inf\left\{\prod_{l=1}^{n}\left(\frac{\underline{\Delta}_{2l-2}(\xi)}{\underline{\Delta}_{2l}(\xi)}\right)^{\beta_l} \,\middle|\, \xi\in\underline{\Xi}\right\}, \tag{2.3}$$

$$\max\left\{\prod_{l=0}^{n} |m_l(Q_l)|^{2\beta_l} \,|\, (Q_0, ..., Q_n)\in\mathscr{Q}_n\right\}$$

$$=\inf\left\{\prod_{l=0}^{n}\left(\frac{\bar{\Delta}_{2l}(\xi)}{\bar{\Delta}_{2l+2}(\xi)}\right)^{\beta_l} \,\middle|\, \xi\in\bar{\Xi}\right\}. \tag{2.4}$$

Moreover, if $(P_1, ..., P_n) \in \mathscr{P}_n$ *and* $\underline{\xi}^* \in \underline{\Xi}$ *are solutions of the problems* (P) *and* (P*) *we have*

$$P_l(x) = \pm \sqrt{\frac{\underline{\Delta}_{2l}(\underline{\xi}^*)}{\underline{\Delta}_{2l-2}(\underline{\xi}^*)}}\, c_l' \underline{M}_{2l}^{-1}(\underline{\xi}^*) f_l(x), \qquad l = 1, ..., n \tag{2.5}$$

where $c_l = (0, ..., 0, 1)' \in \mathbb{R}^{l+1}$. *If* $(Q_0, ..., Q_n) \in \mathscr{Q}_n$ *and* $\bar{\xi}^* \in \bar{\Xi}$ *are solutions of the problems* (Q) *and* (Q*) *we have*

$$Q_l(x) = \pm \sqrt{\frac{\bar{\Delta}_{2l+2}(\bar{\xi}^*)}{\bar{\Delta}_{2l}(\bar{\xi}^*)}}\, c_l' \bar{M}_{2l+2}^{-1}(\bar{\xi}^*) f_l(x), \qquad l = 0, ..., n. \tag{2.6}$$

The polynomials $\{P_l(x)\}_{l=0}^n$ *and* $\{Q_l(x)\}_{l=0}^n$ *are the orthonormal polynomials with respect to the measures* $d\underline{\xi}^*(x)$ *and* $(1 - x^2)\, d\bar{\xi}^*(x)$, *respectively.*

In the following we will show that there always exists a unique solution of the dual problems (P*) and (Q*). The "unique" (up to the sign) solutions of the primal problems (P) and (Q) can then be determined by Theorem 2.1, (2.5) and (2.6). For this task we will need some basic facts of the theory of canonical moments. We will give a brief introduction and state some of the main results which are needed later. The interested reader is referred to the work of Karlin and Shapely [6], Karlin and Studden [7], Skibinsky [15–17], Studden [18–21], and Lau and Studden [8].

For a probability measure ξ on $[-1, 1]$ with moments $c_i = \int_{-1}^{1} x^i\, d\xi(x)$ the canonical moments are defined as follows. For a given set of moments $c_0, c_1, ..., c_{i-1}$ let c_i^+ denote the maximum of the ith moment $\int_{-1}^{1} x^i\, d\eta(x)$ over the set of all probability measures η having the given moments $c_0, c_1, ..., c_{i-1}$. Similarly let c_i^- denote the corresponding minimum. The canonical moments are defined by

$$p_i = \frac{c_i - c_i^-}{c_i^+ - c_i^-}, \qquad i = 1, 2, ...$$

Note that $0 \leqslant p_i \leqslant 1$ and that the canonical moments are left undefined whenever $c_i^+ = c_i^-$. If i is the first index for which this equality holds, then $0 < p_k < 1$, $k = 1, ..., i-2$, p_{i-1} must have the value 0 or 1 and the design ξ is supported at a finite number of points. In this case ξ is the "lower" or "upper principal representation" of its corresponding moment point $(c_0, ..., c_{i-1})$ (see [17, Sect. 1]). As an example consider the Jacobi measure with density $(1 - \alpha)^\alpha (1 + x)^\beta$ $(\alpha > -1, \beta > -1)$. For this measure we have (see [16])

$$p_{2k} = \frac{k}{\alpha + \beta + 2k + 1}, \qquad p_{2k-1} = \frac{\beta + k}{\alpha + \beta + 2k}, \; k \geqslant 1. \tag{2.7}$$

The uniform measure $(\alpha=\beta=0)$ has $p_{2k-1}=1/2$ $(k \geqslant 1)$ and $p_{2k}=k/(2k+1)$ and the arc-sine distribution has $p_k=1/2$ for all k $(\alpha=\beta=-\frac{1}{2})$. The determinants $\underline{\Delta}_{2l}(\xi)$ and $\bar{\Delta}_{2l+2}(\xi)$ appearing in the dual problems can easily be expressed in terms of the canonical moments of the probability measure ξ (see [17] or [20]).

THEOREM 2.2. *Let ξ denote a probability measure with canonical moments $p_1, p_2, ..., q_j=1-p_j$ $(j \geqslant 1)$, $\zeta_0=1$, $\gamma_0=1$, $\zeta_1=p_1$, $\gamma_1=q_1$, $\zeta_j=q_{j-1}p_j$, $\gamma_j=p_{j-1}q_j$ $(j \geqslant 2)$. Then we have*

$$\underline{\Delta}_{2l}(\xi)=2^{l(l+1)} \prod_{i=1}^{l} (\zeta_{2i-1}\zeta_{2i})^{l+1-i},$$

$$\bar{\Delta}_{2l}(\xi)=2^{l(l+1)} \prod_{i=1}^{l} (\gamma_{2i-1}\gamma_{2i})^{l+1-i}.$$

The minimization in (P^*) and (Q^*) can now be easily carried out in terms of the canonical moments and we obtain the following result.

THEOREM 2.3. *Let $\sigma_i=\sum_{l=i}^{n} \beta_l$ then we have the following:*

(a) *The solution $\underline{\xi}^*$ of the dual problem (P^*) is unique and has canonical moments*

$$\begin{aligned} p_{2i-1}&=\frac{1}{2}, && i=1, ..., n \\ p_{2i}&=\frac{\sigma_i}{\sigma_i+\sigma_{i+1}}, && i=1, ..., n-1 \\ p_{2n}&=1. \end{aligned} \tag{2.8}$$

(b) *The solution $\bar{\xi}^*$ of the dual problem (Q^*) is unique and has canonical moments*

$$\begin{aligned} p_{2i-1}&=\frac{1}{2}, && i=1, ..., n+1 \\ p_{2i}&=\frac{\sigma_i}{\sigma_{i-1}+\sigma_i}, && i=1, ..., n \\ p_{2n+2}&=0. \end{aligned} \tag{2.9}$$

Proof. We give a proof of (a); the second case is treated in the same way. Minimizing

$$\prod_{l=1}^{n}\left(\frac{\Delta_{2l-2}(\xi)}{\Delta_{2l}(\xi)}\right)^{\beta_l}=\prod_{l=1}^{n}\left[2^{2l}\prod_{j=1}^{l}(\zeta_{2j-1}\zeta_{2j})\right]^{\beta_l}$$

(here the last identity results from Theorem 2.2) in terms of the canonical moments we obtain by straightforward algebra the moments in (2.8). The fact that ξ^* is the unique design with canonical moments (2.8) results from $p_{2n}=1$ (see [17]). ∎

Theorem 2.3 provides essentially a complete solution of the dual problems (P*) and (Q*) and the (unique) solutions of the primal problems (P) and (Q) can be obtained by Theorem 2.1. Instead of applying the formulas (2.5) and (2.6) directly, it is convenient to use the orthonormality property of the polynomials $\{P_l(x)\}_{l=1}^{n}$ and $\{Q_l(x)\}_{l=0}^{n}$ with respect to the minimizing measures and the following lemma (see [20]).

LEMMA 2.4. *Let ξ denote a probability measure on the interval $[-1, 1]$ with canonical moments $p_1, p_2, ..., \zeta_1=p_1, \gamma_1=q_1, \zeta_j=q_{j-1}p_j, \gamma_j=p_{j-1}q_j$ $(j\geqslant 2,\ q_j=1-p_j)$ and let the polynomials $\tilde{P}_l$ and $\tilde{Q}_l$ be defined recursively by $(\tilde{P}_{-1}=\tilde{Q}_{-1}=0,\ \tilde{P}_0=\tilde{Q}_0=1)$*

$$\begin{aligned}\tilde{P}_{j+1}(x)&=(x+1-2(\zeta_{2j}+\zeta_{2j+1}))\,\tilde{P}_j(x)\\&\quad-4\zeta_{2j-1}\zeta_{2j}\tilde{P}_{j-1}(x)\qquad(j\geqslant 0)\\ \tilde{Q}_{j+1}(x)&=(x+1-2(\gamma_{2j+2}+\gamma_{2j+3}))\,\tilde{Q}_j(x)\\&\quad-4\gamma_{2j+1}\gamma_{2j+2}\tilde{Q}_{j-1}(x)\qquad(j\geqslant 0).\end{aligned}$$

The polynomials $\{\tilde{P}_l(x)\}_{l=1}^{n}$ and $\{\tilde{Q}_l(x)\}_{l=0}^{n}$ are orthogonal with respect to the measures $d\xi(x)$ and $(1-x^2)\,d\xi(x)$, respectively. Moreover, the L_2-norms of the polynomials with respect to these measures are given by

$$\int_{-1}^{1}\tilde{P}_l^2(x)\,d\xi(x)=2^{2l}\prod_{j=1}^{l}\zeta_{2j-1}\zeta_{2j}=\frac{\Delta_{2l}(\xi)}{\Delta_{2l-2}(\xi)}\tag{2.10}$$

$$\int_{-1}^{1}\tilde{Q}_l^2(x)(1-x^2)\,d\xi(x)=2^{2(l+1)}\prod_{j=1}^{l+1}\gamma_{2j-1}\gamma_{2j}=\frac{\bar{\Delta}_{2l+2}(\xi)}{\bar{\Delta}_{2l}(\xi)}.\tag{2.11}$$

3. SOLUTIONS OF THE DUAL PROBLEM WITH SIMPLE STRUCTURE

For a general set of weights $\{\beta_l\}$ it seems to be difficult to identify the underlying measure (i.e., the solution of the dual problem) corresponding to an "optimal" set of canonical moments and to derive explicit expressions for the orthonormal polynomials with respect to these measures. In this section we are considering a one-parameter class of weights which yields explicit expressions for the solutions of the primal and dual problems (P), (Q), (P*), and (Q*). To this end we define for $z \geq 0$ the weights

$$\beta_l(z) = z \frac{\Gamma(n)\,\Gamma(n+z-l)}{\Gamma(n+z)\,\Gamma(n+1-l)}, \qquad l = 1, ..., n \tag{3.1}$$

for the extremal problem (P) and

$$\beta_l(z) = z \frac{\Gamma(n+1)\,\Gamma(n+z-l)}{\Gamma(n+z+1)\,\Gamma(n+1-l)}, \qquad l = 0, ..., n \tag{3.2}$$

for the problem (Q). Here $\Gamma(z)$ denotes the gamma-function and the case $z = 0$ is understood as the limit $\lim_{z \to 0}(\beta_0(z), ..., \beta_n(z)) = (0, ..., 0, 1)$. Using the identity ($n \in \mathbb{N}_0$, $z \in \mathbb{R} \backslash \{0, -1, -2, ...\}$)

$$\sum_{l=i}^{n} \frac{\Gamma(n+z-l)}{\Gamma(n+1-l)} = \frac{\Gamma(n+z-i+1)}{\Gamma(n-i+1)} \frac{1}{z} \tag{3.3}$$

(which follows by an induction argument) it can easily be shown that $\sum_l \beta_l(z) = 1$. Note that the class defined by (3.1) (and (3.2)) includes the important case of uniform weights $\beta_l = 1/n$ ($\beta_l = 1/(n+1)$) which is obtained for the case $z = 1$. By Theorem 2.3 we have that the solution ξ^* of the dual problem (P*) is characterized by the canonical moments $p_{2i-1} = 1/2$ ($i = 1, ..., n$), $p_{2n} = 1$, and

$$p_{2i} = \frac{\sigma_i}{\sigma_i + \sigma_{i+1}} = \frac{n-i+z}{z+2(n-i)}, \qquad i = 1, ..., n-1, \tag{3.4}$$

where the last identity follows again from (3.3) and straightforward algebra. Similarly the canonical moments of the solution $\bar{\xi}^*$ of (Q*) are given by $p_{2i-1} = 1/2$ ($i = 1, ..., n+1$), $p_{2n+2} = 0$, and

$$p_{2i} = \frac{\sigma_i}{\sigma_{i-1} + \sigma_i} = \frac{n-i+1}{z+2(n-i+1)}, \qquad i = 1, ..., n. \tag{3.5}$$

In these cases an explicit representation of the minimizing measures in the dual problems is possible and was obtained by Dette [2]. In what follows $C_n^{(\alpha)}(x)$ denotes the nth ultraspherical polynomial orthogonal with respect to the measure $(1-x^2)^{\alpha - 1/2}\,dx$ with leading coefficient $2^n \Gamma(\alpha + n)/\Gamma(n+1)\,\Gamma(\alpha)$ (see [22, p. 93]).

THEOREM 3.1. (a) *The solution* $\underline{\xi}^*$ *of the dual problem* (P^*) *for the weights* $\beta_l(z)$ *defined by* (3.1) *is supported at the zeros of the polynomial* $(1-x^2)\,C_{n-1}^{(z/2+1)}(x)$ *and the masses at all interior support points are equal to* $1/(n+z)$. *The masses at the boundary points* ± 1 *are equal to* $(z+1)/2$ *times the masses at the interior points.*

(b) *The solution* $\bar{\xi}^*$ *of the dual problem* (Q^*) *for the weights* $\beta_l(z)$ *defined by* (3.2) *puts equal masses at the zeros of the polynomial* $C_{n+1}^{(z/2)}(x)$.

The simple form of the solutions of the dual problems suggests that it is also possible to obtain explicit representations for the solutions of the primal problems (P) and (Q) if the weights are defined by (3.1) and (3.2), respectively. By Theorem 2.1 the maximizing polynomials are orthonormal with respect to the measures $d\underline{\xi}^*(x)$ and $(1-x^2)\,d\bar{\xi}^*(x)$. The following result shows that these polynomials are weighted sums of ultraspherical polynomials.

THEOREM 3.2. (a) *The solution* $(P_1, ..., P_n)$ *of the primal problem* (P) *for the weights defined in* (3.1) *is given by*

$$P_l(x) = \pm\alpha(n, l, z) \cdot \sum_{j=0}^{\lfloor l/2 \rfloor} (-1)^j \beta(n, l, z, j)\, C_{l-2j}^{(z/2+n-l+j)}(x), \qquad l = 1, ..., n.$$

(b) *The solution* $(Q_0, ..., Q_n)$ *of the primal problem* (Q) *for the weights defined in* (3.2) *is given by*

$$Q_l(x) = \pm\alpha(n+1, l+1, z) \cdot \sum_{j=0}^{\lfloor l/2 \rfloor} (-1)^j \gamma(n, l, z, j)\, C_{l-2j}^{(z/2+n-l+j+1)}(x),$$

$$l = 0, ..., n.$$

Here the numbers $\alpha(n, l, z)$, $\beta(n, l, z, j)$, $\gamma(n, l, z, j)$ *are defined by* $\alpha(n, 0, z) = \beta(n, 0, z, 0) = \gamma(n, 0, z, 0) = 1$ *and*

$$\alpha(n, l, j) = \left[\frac{z + 2(n-l)}{\Gamma(n)\,\Gamma(n+z)\,\Gamma(n-l+z)\,\Gamma(n-l+1)}\right]^{1/2}$$

$$\beta(n, l, z, j) = [jz + l(n-l+j)]\,\frac{\Gamma(l-j)\,\Gamma(n-l+z+j)\,\Gamma(n-l+j)}{\Gamma(j+1)[z+2(n-l+j)]}$$

$$\gamma(n, l, z, j) = \frac{\Gamma(l+1-j)}{\Gamma(j+1)}\,\Gamma(n-l+j+1)\,\Gamma(n-l+z+j).$$

Proof. We will give a detailed proof of part (b) and sketch the main steps of the proof of (a) very briefly. For simplification of the notation we introduce the continuants

$$K\begin{pmatrix} & a_1 & & \cdots & a_n & \\ b_0 & & b_1 & \cdots & & b_n \end{pmatrix} = \det\begin{bmatrix} b_0 & -1 & & \\ a_1 & b_1 & -1 & \\ & & \ddots & -1 \\ & & a_n & b_n \end{bmatrix}$$

(all other elements in the matrix are 0) which have various applications in the theory of continued fractions (see, e.g., Wall [23]). By Lemma 2.4 the orthogonal polynomials with respect to the measure $(1-x^2)\,d\xi^*(x)$ with leading coefficient 1 are recursively given by ($\tilde{Q}_0(x)=1$, $\tilde{Q}_1(x)=x$)

$$\tilde{Q}_l(x)=\tilde{Q}_{l-1}(x)-p_{2l-2}q_{2l}\tilde{Q}_{l-2}(x), \qquad l\geqslant 2, \tag{3.6}$$

where the canonical moments p_{2l} are defined by (3.5) (note that the canonical moments of odd order of ξ^* are 1/2) and it can easily be checked that the polynomials $\tilde{Q}_l(x)$ can also be written as the continuant ($l\geqslant 2$)

$$\tilde{Q}_l(x)=K\begin{pmatrix} & -\dfrac{n(n-1+z)}{(z+2n)(z+2(n-1))} & & \cdots & -\dfrac{(n-l+2)(n-l+z+1)}{(z+2(n-l+1))(z+2(n-l+2))} & \\ x & & x & \cdots & x & x \end{pmatrix}.$$

To obtain an explicit representation of $\tilde{Q}_l(x)$ in terms of ultraspherical polynomials we need the following lemma whose proof is deferred to Appendix B.

LEMMA 3.3. *Let $k\geqslant 3$, $w+2a>-3$. Then the polynomial*

$$G_{k-1}^{(a,w)}(x)=K\begin{pmatrix} & -\dfrac{(k-2+a)(k-1+a+w)}{(w+2(k-2+a))(w+2(k-1+a))} & & \cdots & -\dfrac{(a+1)(a+2+w)}{(w+2(a+1))(w+2(a+2))} & \\ x & & x & \cdots & x & x \end{pmatrix}.$$

$(G_0^{(a,w)}(x)=1,\ G_1^{(a,w)}(x)=x)$ *is given by*

$$G_{k-1}^{(a,w)}(x)=2^{-(k-1)}\frac{\Gamma((w+2)/2+a)\,\Gamma(k)}{\Gamma(w/2+a+k)}$$
$$\times\sum_{j=0}^{\lfloor (k-1)/2\rfloor}(-1)^j\frac{\Gamma(a+j)}{\Gamma(a)\,\Gamma(j+1)}\,\frac{\Gamma(w+a+1+j)}{\Gamma(w+a+1)}$$
$$\times\frac{\Gamma(k-j)}{\Gamma(k)}\,C_{k-2j-1}^{(w/2+1+a+j)}(x).$$

Using Lemma 2.4 and (3.5) it follows by straightforward calculations for the L_2-norms of the orthogonal polynomials $\tilde{Q}_l(x)$ with respect to the measure $(1-x^2)\,d\bar{\xi}^*(x)$ that

$$\delta_l^2=\int_{-1}^{1}\tilde{Q}_l^2(x)(1-x^2)\,d\bar{\xi}^*(x)=\prod_{j=1}^{l+1}p_{2j-2}q_{2j}$$
$$=\frac{\Gamma(n+1)\,\Gamma(n+z+1)}{\Gamma(n+1-l)\,\Gamma(n-l+z)(z+2(n-l))}\left[\frac{\Gamma(z/2+1+n-l)}{\Gamma(z/2+n+1)\,2^l}\right]^2.$$

Thus we obtain from Lemma 3.3 $(a=n-l+1,\ k=l+1,\ w=z-2)$ for the orthonormal polynomials with respect to the measure $(1-x^2)\,d\bar{\xi}^*(x)$

$$Q_l(x)=\frac{\tilde{Q}_l(x)}{\delta_l}$$
$$=\alpha(n+1,l+1,z)\cdot\sum_{j=0}^{\lfloor l/2\rfloor}(-1)^j\gamma(n,l,z,j)\,C_{l-2j}^{(z/2+n-l+j+1)}(x)$$

$(l=0,\ldots,n)$ which completes the proof of part (b). For the proof of (a) we remark that by Lemma 2.4 and (3.4) the orthogonal polynomials $\tilde{P}_l(x)$ with respect to the measure $d\underline{\xi}^*(x)$ can be written as $(\tilde{P}_0(x)=1,\ \tilde{P}_1(x)=x)$

$$\tilde{P}_l(x)=K\left(\begin{array}{ccc} & -\dfrac{n-1+z}{z+2(n-1)}-\dfrac{(n-1)(n-2+z)}{(z+2(n-1))(z+2(n-2))} & \\ x & & x \\ & \cdots\qquad -\dfrac{(n-l+1+z)(n-l+2)}{(z+2(n-l+1))(z+2(n-l+2))} & \\ & \cdots\quad x & x \end{array}\right).$$

$$=xG_{l-1}^{(n-l+1,z-2)}(x)-\frac{n-1+z}{z+2(n-1)}\,G_{l-2}^{(n-l+1,z-2)}(x),$$

where in the last step we have used an expansion of the determinant and the definition of the polynomials $G_{k-1}^{(a,w)}(x)$. The L_2-norms of the polynomials $\tilde{P}_l(x)$ with respect to the measure $d\xi^*(x)$ are given by

$$\int_{-1}^{1} \tilde{P}_l(x)^2 \, d\xi^*(x) = \frac{\Gamma(n)\,\Gamma(n+z)}{\Gamma(n-l+z)\,\Gamma(n-l+1)(z+2(n-l))} \times \left[\frac{\Gamma(z/2+1+n-l)}{\Gamma(z/2+n)\,2^{l-1}}\right]^2$$

and the assertion (a) of Theorem 3.2 now follows from Lemma 3.3 and straightforward but tedious algebra. ∎

EXAMPLE 3.4. In the case $z=0$ we have for the weights $\beta_n(0)=1$, $\beta_l(0)=0$ if $l \leqslant n-1$ and we obtain the extremal problems for the Cheybshev polynomials described in the Introduction. In this case the canonical moments in (3.4) and (3.5) are all 1/2 and the corresponding probability measure is the discrete arc-sine distribution. Taking the limit $z \to 0$ it can easily be shown that the polynomials $P_n(x)$ and $Q_n(x)$ of Theorem 3.2 reduce to the Chebyshev polynomials $T_n(x)$ and $U_n(x)$, respectively.

EXAMPLE 3.5. Taking $z=1$ we obtain that all weights β_l in the extremal problems (P) and (Q) are equal. If $n=3$ the problem (P) is to maximize the product of the absolute values of the highest coefficients $\prod_{l=1}^{3} |m_l(P_l)|$ where the polynomials P_l (of degree l) satisfy

$$P_1^2(x)+P_2^2(x)+P_3^2(x) \leqslant 3 \qquad \text{for all} \quad x \in [-1, 1].$$

By an application of Theorem 3.2(a) it is now straightforward to show that this product is maximized for the polynomials

$$P_1(x)=\sqrt{\frac{5}{3}}\,x, \qquad P_2(x)=\frac{5}{2}\left(x^2-\frac{3}{5}\right), \qquad P_3(x)=\frac{5}{2}\sqrt{3}\left(x^3-\frac{13}{15}x\right)$$

and it follows from Theorem 2.1 and Theorem 3.1(a) that these polynomials are the orthonormal polynomials with respect to the measure ξ^* which puts equal masses at the points -1, $-1/\sqrt{5}$, $1/\sqrt{5}$, and 1 (the zeros of $(1-x^2)\,C_2^{(3/2)}(x)$). This measure is the solution of the D-optimal design problem for the cubic regression model (see [7, p. 339]). In the same way we have by Theorem 3.2(b) that under the restriction

$$(1-x^2)\,[Q_0^2(x)+Q_1^2(x)+Q_2^2(x)+Q_3^2(x)] \leqslant 4 \qquad \text{for all} \quad x \in [-1, 1]$$

the product $\prod_{l=0}^{3}|m_l(Q_l)|$ is maximized for the polynomials

$$Q_0(x)=\frac{\sqrt{7}}{2}, \qquad Q_1(x)=\frac{7\sqrt{5}}{6}x,$$

$$Q_2(x)=\frac{35\sqrt{3}}{12}\left(x^2-\frac{9}{35}\right), \qquad Q_3(x)=\frac{105}{12}\left(x^3-\frac{11}{21}x\right).$$

4. An Extremal Property of the Ultraspherical Polynomials

In Section 3 we investigated weights β_l which yield solutions of the dual problems (P*) and (Q*) with a very simple structure. The solutions of the primal problems are more complicated (see Theorems 3.1 and 3.2). In this section we will aim for a simple structure of the solutions of the primal problems (P) and (Q). We will investigate a class of weights which yield the ultraspherical polynomials as extremal polynomials. To this end define.

$$\beta_l(\alpha)=\begin{cases} 2\alpha\dfrac{\Gamma(l+1)\,\Gamma(2\alpha+1)}{\Gamma(l+2\alpha+2)} & \text{if} \quad 0\leqslant l\leqslant n-1 \\ \dfrac{\Gamma(n+1)\,\Gamma(2\alpha+1)}{\Gamma(n+2\alpha+1)} & \text{if} \quad l=n, \end{cases} \tag{4.1}$$

where $\alpha\geqslant 0$. Then we have the following theorem.

Theorem 4.1. *Let $\alpha\geqslant 0$ and $\beta_l(\alpha)$ be defined by (4.1). Among all polynomials $Q_0(x), ..., Q_n(x)$ (of degree $0, ..., n$) satisfying*

$$(1-x^2)\left\{4\alpha\sum_{l=0}^{n-1}(l+\alpha+1)\,Q_l(x)^2+[(2\alpha+n+1)\,Q_n(x)]^2\right\}\leqslant 1$$

for all $x\in[-1,1]$, the "weighted" product of the absolute values of the highest coefficients $\prod_{l=0}^{n}|m_l(Q_l)|^{\beta_l(\alpha)}$ is maximized for the polynomials (proportional to the ultraspherical polynomials) $Q_l(x)=B(2\alpha+1,l+1)\,C_l^{(\alpha+1)}(x)$.

Proof. We consider the extremal problem (Q) for the weights $\beta_l(\alpha)$ defined by (4.1). By Theorem 2.3 the solution ξ^* of the dual problem (Q*) has canonical moments $p_{2l-1}=1/2$ $(l=1, ..., n+1)$, $p_{2n+2}=0$, and

$$p_{2i}=\frac{\sigma_i}{\sigma_{i-1}+\sigma_i}=\frac{i}{2(i+\alpha)}, \qquad i=1, ..., n,$$

where the last identity follows by induction and from the representation $\sigma_{i-1} = \beta_{i-1}(q_{2i}/(q_{2i} - p_{2i}))$ (this is an immediate consequence of (2.9)). From Theorem 2.1 we have that the maximizing polynomials are orthogonal with respect to the measure $(1-x^2)\, d\xi^*(x)$. Because the canonical moments of ξ^* coincide with the first $2n+1$ canonical moments of the Jacobi measure in (2.7) ($\alpha = \beta = \alpha - 1/2$) the polynomials $Q_l(x)$ are proportional to the ultraspherical polynomials $C_l^{(\alpha+1)}(x)$. Comparing the leading coefficients of these polynomials we obtain

$$(Q_l(x))^2 = 2\,\frac{\Gamma(l+1)\,\Gamma(2\alpha+1)}{\Gamma(l+2+2\alpha)}\,(l+\alpha+1)(C_l^{(\alpha+1)}(x))^2,$$

$$l = 0, ..., n-1$$

$$(Q_n(x))^2 = \frac{\Gamma(n+1)\,\Gamma(2\alpha+1)}{\Gamma(n+1+2\alpha)}\,(C_n^{(\alpha+1)}(x))^2,$$

and the assertion now follows by a simple transformation of the normalizing condition in the extremal problem (Q). ∎

EXAMPLE 4.2. For $\alpha = 0$ we obtain the extremal property of the Chebyshev polynomials of the second kind described in the Introduction (note that $C_n^{(1)}(x) = U_n(x)$). To give a "non-trivial" example we consider the case $\alpha = 1/2$. Then it follows that among all polynomials satisfying

$$(1-x^2)\left\{\sum_{l=0}^{n-1} (2l+3)\, Q_l^2(x) + [(n+2)\, Q_n(x)]^2\right\}$$

$$\leqslant 1 \qquad \text{for all} \quad x \in [-1, 1]$$

the weighted product of the absolute values of the highest coefficients

$$\prod_{l=0}^{n-1} |m_l(Q_l)|^{1/(l+1)(l+2)} \cdot |m_n(Q_n)|^{1/(n+1)}$$

is maximized for the polynomials $[(l+1)(l+2)]^{-1}\, C_l^{(3/2)}(x)$ $(l = 0, ..., n)$.

Note that Theorem 4.1 shows that the ultraspherical polynomials $C_l^{(\alpha)}(x)$ satisfy an extremal property whenever $\alpha \geqslant 1$. In a similar way we can derive an analogous result (using the primal problem (P)) for the case $-1/2 < \alpha \leqslant 0$. Its proof is performed in the same way as the proof of Theorem 4.1 and therefore omitted.

THEOREM 4.3. *Let* $-1/2<\alpha\leqslant 0$ *and*

$$\beta_l(\alpha)=\begin{cases} -2\alpha\dfrac{\Gamma(l+2\alpha)}{\Gamma(l+1)\,\Gamma(2\alpha+1)} & \text{if } 1\leqslant l\leqslant n-1 \\ \dfrac{\Gamma(n+2\alpha)}{\Gamma(n)\,\Gamma(2\alpha+1)} & \text{if } l=n. \end{cases}$$

Among all polynomials $P_1(x), ..., P_n(x)$ *of degree* $1, ..., n$ *satisfying*

$$-4\alpha\cdot\sum_{l=1}^{n-1}\frac{l+\alpha}{l^2}P_l^2(x)+P_n^2(x)\leqslant 1$$

for all $x\in[-1,1]$, *the "weighted" product of the absolute values of the highest coefficients* $\prod_{l=1}^{n}|m_l(P_l)|^{\beta_l(\alpha)}$ *is maximized for the polynomials (proportional to the ultraspherical polynomials)* $P_l(x)=(l/2\alpha)\,C_l^{(\alpha)}(x)$.

Remark 4.4. Note that the restrictions in Theorems 4.1 and 4.3 can easily be transformed in such a way that the extremal polynomials are exactly the ultraspherical polynomials. Thus these theorems give extremal properties for the ultraspherical polynomials $C_l^{(\alpha)}(x)$ provided that the parameter α is nonnegative (Theorem 4.1) or nonpositive (Theorem 4.3). This results from the fact that the weights in the extremal problems (P) and (Q) have to be nonnegative. For negative weights β_l the duality in Theorem 2.1 does not hold any longer and thus Theorem 4.2 and Theorem 4.3 are not true if $\alpha<0$ or $\alpha>0$, respectively. In these cases counterexamples can easily be constructed.

5. GENERALIZATIONS

In this section we will consider weights $\beta_1, ..., \beta_n$ in the extremal problems (P) and (Q) where most of these weights β_l are vanishing. More precisely, for $n=kr$ ($k, r\in\mathbb{N}$) we will consider weights of the form

$$\beta_j\begin{cases} :=\beta_l^*\geqslant 0 & \text{if } j=kl,\ l=1, ..., r \\ :=0 & \text{else} \end{cases} \tag{5.1}$$

for the primal problem (P) and weights of the form

$$\beta_j\begin{cases} :=\beta_l^*\geqslant 0 & \text{if } j=k(l+1)-1,\ l=0, ..., r \\ :=0 & \text{else} \end{cases} \tag{5.2}$$

for the primal problem (Q) ($\sum_j \beta_j = \sum_l \beta_l^* = 1$). For example, if $k \in \mathbb{N}$ and $r = 2$ the problem (P) for the weights in (5.1) is to maximize the product of the absolute values of the highest coefficients $|m_k(P_k)|^{1-\beta} |m_{2k}(P_{2k})|^{\beta}$ of two polynomials P_k and P_{2k} of degree k and $2k$ subject to the restriction

$$(1-\beta) P_k^2(x) + \beta P_{2k}^2(x) \leqslant 1 \qquad \text{for all} \quad x \in [-1, 1] \tag{5.3}$$

($(\beta_1, \beta_2, \beta_3, \beta_4) = (0, \beta_1^*, 0, \beta_2^*) =: (0, 1-\beta, 0, \beta)$). The following theorem shows that the solution of this problem is given by $P_1(T_k(x))$ and $P_2(T_k(x))$ where P_1 and P_2 are the linear and quadratic polynomials maximizing the corresponding product subject to the restriction

$$(1-\beta) P_1^2(x) + \beta P_2^2(x) \leqslant 1 \qquad \text{for all} \quad x \in [-1, 1]$$

and $T_k(x)$ is the kth Chebyshev polynomial of the first kind.

THEOREM 5.1. (a) *The maximizing polynomials in the problem* (P) *for the weights* β_j *defined by* (5.1) *are given by* $\{P_l^*(T_k(x))\}_{l=1}^r$ *where the polynomials* $\{P_l^*(x)\}_{l=1}^r$ *are the solution of the primal problem* (P) *for the weights* $\beta_1^*, ..., \beta_r^*$ $(n = r)$.

(b) *The maximizing polynomials in the problem* (Q) *for the weights* β_j *defined by* (5.2) *are given by* $\{U_{k-1}(x) Q_l^*(T_k(x))\}_{l=0}^r$ *where the polynomials* $\{Q_l^*(x)\}_{l=0}^r$ *are the solution of the primal problem* (Q) *for the weights* $\beta_0^*, ..., \beta_r^*$ $(n = r)$.

Proof. We will give a proof for (b); the other part is treated in the same way. Because the case $k = 1$ is trivial we assume $k \geqslant 2$ throughout this proof. By an application of Theorem 2.3 we obtain for the canonical moments of the solution $\bar{\xi}_\beta^*$ of the dual problem (Q*) for the weights β_j defined by (5.2)

$$\begin{aligned} p_{2kj} &= \frac{\sigma_j^*}{\sigma_{j-1}^* + \sigma_j^*} := p_{2j}^*, && j = 1, ..., r \\ p_i &= \frac{1}{2}, && i \leqslant 2k(r+1) - 1,\ i \neq 2kj \\ p_{2k(r+1)} &= 0, \end{aligned} \tag{5.4}$$

where $\sigma_j^* := \sigma_{kj} = \sum_{l=kj}^{kr} \beta_l = \sum_{l=j}^{r} \beta_l^*$. By Theorem 2.1 we have to calculate the orthogonal polynomials $Q_{k-1}(x)$, $Q_{2k-1}(x)$, ... with respect to the measure $(1-x^2)\, d\bar{\xi}_\beta^*(x)$ (with leading coefficient 1) which are given by

$$\tilde{Q}_{kl-1}(x) = K\begin{pmatrix} \frac{-1}{4} & \cdots & \frac{-1}{4} & \frac{-p^*_{2l-2}}{2} & \frac{-q^*_{2l-2}}{2} & \frac{-1}{4} \\ x & \cdots & & x & x & \end{pmatrix.$$
$$\begin{matrix} \cdots & \frac{-1}{4} & \frac{-p^*_2}{2} & \frac{-q^*_2}{2} & \frac{-1}{4} & \cdots & \frac{-1}{4} \\ \cdots & & x & x & & \cdots & x \end{matrix}\Bigg).$$

Interpreting $Q_{kl-1}(x)$ as the polynomial in the denominator of a continued fraction we obtain by a contraction (such that the convergents of the transformed continued fractions attain successively the values of the $(k-1)$st, $(2k-1)$st, ... convergents of the original one (see Perron [10, Band II, pp. 11–12])

$$\begin{aligned} Q_{kl-1}(x) &= K\begin{pmatrix} -a^3 U_{k-1}(x)\, q^*_{2l-2} p^*_{2l-4} & -a^2 q^*_{2l-4} p^*_{2l-6} & \cdots & -a^2 q^*_4 p^*_2 \\ a^2 T_k(x)\, U_{k-1}(x) & a T_k(x) & \cdots & a T_k(x) \end{pmatrix} \\ &= \left(\frac{1}{2}\right)^{(k-1)l} U_{k-1}(x) \\ &\quad \cdot K\begin{pmatrix} -q^*_{2l-2} p^*_{2l-4} & -q^*_{2l-4} p^*_{2l-6} & \cdots & -q^*_4 p^*_2 \\ T_k(x) & T_k(x) & \cdots & T_k(x) \end{pmatrix} \\ &= \left(\frac{1}{2}\right)^{(k-1)l} U_{k-1}(x) \cdot Q^*_{l-1}(T_k(x)). \end{aligned}$$

Here we have used the notation $a = (1/2)^{k-1}$, the recursive definition (see (3.6)) of the polynomials $\{Q^*_l\}_{l=0}^{r}$ ($Q^*_0(x) = 1$, $Q^*_1(x) = x$)

$$Q^*_{l+1}(x) = xQ^*_l(x) - q^*_{2l+2} p^*_{2l} Q^*_{l-1}(x), \qquad l \geqslant 1, \tag{5.5}$$

and have applied the representation for the Chebyshev polynomials of the second kind

$$U_k(x) = 2^k \cdot K\begin{pmatrix} & -\frac{1}{4} & \cdots & -\frac{1}{4} & \\ x & x & \cdots & & x \end{pmatrix}.$$

Observing (5.4), (5.5), (2.9), Theorem 2.3, and Lemma 2.4 we see that the polynomials Q^*_l are orthogonal with respect to the measure $(1-x^2)\, d\bar{\xi}^*_{\beta^*}(x)$ where $\bar{\xi}^*_{\beta^*}$ is the solution of the dual problem (Q*) for the weights $\beta_0^*, ..., \beta_r^*$ $(n=r)$. The assertion of the theorem now follows from Theorem 2.1 by a calculation of the normalizing constants (using Lemma 2.4) which transform the polynomials $Q_{kl-1}(x)$ and $Q_{l-1}^*(x)$ into the orthonormal polynomials with respect to the measures $(1-x^2)\, d\xi_\beta^*(x)$ and $(1-x^2)\, d\xi_{\beta^*}^*(x)$. ∎

COROLLARY 5.2. *Let $k \in \mathbb{N}$ and $\beta \in (0, 1]$.*

(a) *Among all polynomials $P_k(x)$ and $P_{2k}(x)$ (of degree k and $2k$) satisfying (5.3) the product of the highest coefficients $|m_k(P_k)|^{1-\beta}$ $|m_{2k}(P_{2k})|^\beta$ is maximized for the polynomials*

$$P_k(x) = \pm\sqrt{1+\beta}\, T_k(x) \qquad \textit{and} \qquad P_{2k}(x) = \pm\frac{1}{\sqrt{\beta}}[(1+\beta)\, T_k^2(x) - 1].$$

(b) *Among all polynomials $Q_{k-1}(x)$ and $Q_{2k-1}(x)$ (of degree $k-1$ and $2k-1$) satisfying*

$$(1-x^2)\,[(1-\beta)\, Q_{k-1}^2 + \beta Q_{2k-1}^2(x)] \leqslant 1 \qquad \textit{for all} \quad x \in [-1, 1]$$

the product of the highest coefficients $|m_{k-1}(Q_{k-1})|^{1-\beta}\, |m_{2k-1}(Q_{2k-1})|^\beta$ is maximized for the polynomials

$$Q_{k-1}(x) = \pm\sqrt{1+\beta}\, U_{k-1}(x) \qquad \textit{and} \qquad Q_{2k-1}(x) = \pm\frac{1+\beta}{2\sqrt{\beta}}\, U_{2k-1}(x).$$

Proof. We will give a proof of (a); part (b) is treated in the same way. By Theorem 5.1 we have to solve the problem (P) for $n=r=2$ and a linear and quadratic polynomial. The canonical moments of the solution of the dual problem are obtained from (2.8) as $p_1 = p_3 = 1/2$, $p_4 = 1$, and $p_2 = 1/(1+\beta)$ and the assertion now follows directly from Theorem 2.1, Lemma 2.4, and Theorem 5.1. ∎

Remark 5.3. Note that Theorem 5.1 yields some interesting results generalizing the theorems of Sections 3 and 4. To give an example we consider the problem (P) for the weights β_j defined by (5.1) where the weights β_l^* in (5.1) are given by (3.1). In this case it follows from Theorem 5.1 that the maximizing polynomials have the representation

$$P_{kl}(x) = \pm\alpha(n, l, z) \cdot \sum_{j=0}^{\lfloor l/2 \rfloor} (-1)^j \beta(n, l, z, j)$$
$$\cdot\, C_{l-2j}^{(z/2+n-l+j)}(T_k(x)), \qquad l = 1, ..., n,$$

where the numbers $\alpha(\cdot)$ and $\beta(\cdot)$ are defined as in Theorem 3.2. A similar result can also be obtained for the solution of the dual problem (P*) (see Theorem 3.1(a)).

APPENDIX A: PROOF OF THEOREM 2.1

EXAMPLE. The proof of Theorem 2.1 involves general arguments of convex analysis as given in Roberts and Varberg [13] or Rockafellar [14]. For a better understanding of the difficult and technical proof of the duality we start proving Theorem 2.1 for the special case $\beta_1 = \cdots = \beta_{n-1} = 0$, $\beta_n = 1$ where the situation is more transparent. Observing that $\Delta_{2l-2}(\xi)/\Delta_{2l}(\xi) = c_l' \underline{M}_{2l}^{-1}(\xi)\, c_l$ we obtain for the dual problem (using Cauchy's inequality)

$$\begin{aligned}
\inf_{\xi} c_n' \underline{M}_{2n}^{-1}(\xi)\, c_n &= \inf_{\xi} \sup_{d \neq 0} \frac{(c_n' d)^2}{d' \underline{M}_{2n}(\xi)\, d} \\
&= \left[\sup_{\xi} \inf_{c_n' d = 1} d' \underline{M}_{2n}(\xi)\, d \right]^{-1} \\
&= \left[\inf_{c_n' d = 1} \sup_{\xi} d' \underline{M}_{2n}(\xi)\, d \right]^{-1} \\
&= \sup_{d \neq 0} \frac{(c_n' d)^2}{\sup_{\xi} \int_{-1}^{1} (d' f_n(x))^2 \, d\xi(x)} \\
&= \sup_{d \neq 0} \frac{(c_n' d)^2}{\sup_{x \in [-1,1]} |d' f_n(x)|^2},
\end{aligned}$$

where the second line follows from the first one by standard arguments of game theory (see, e.g., [13, p. 131]). This yields

$$\begin{aligned}
\inf_{\xi} c_n' \underline{M}_{2n}^{-1}(\xi)\, c_n &= \sup\{ (c_n' d)^2 \mid \sup_{x \in [-1,1]} |d' f_n(x)|^2 = 1 \} \\
&= \sup\{ |m_n(P_n)|^2 \mid \max_{x \in [-1,1]} |P_n(x)| \leqslant 1 \}
\end{aligned}$$

and establishes the duality in Theorem 2.1 for the special case $\beta_1 = \cdots = \beta_{n-1} = 0$, $\beta_n = 1$. The representation (2.5) now follows by discussing the case of equality in Cauchy's inequality in the first line.

The general proof of Theorem 2.1 is in essence a generalization of a duality result of Pukelsheim [11] and is discussed in the more general situation of model robust designs in Dette [4]. For completeness and according to a comment of a referee we give detailed proofs of the essential steps for the situation considered in this paper. To treat both cases of Theorem 2.1 in one proof we introduce the following notation. Let $g_0(x), ..., g_n(x)$ denote $n+1$ continuous linearly independent functions defined on the interval $[-1, 1]$, $c_l = (0, ..., 0, 1)' \in \mathbb{R}^{l+1}$ and define for a probability measure ξ on $[-1, 1]$

$$A_l(\xi) = \int_{-1}^{1} f_l(x) f_l'(x) \, d\xi(x) \in \mathbb{R}^{(l+1)\times(l+1)}, \tag{A.1}$$

where $f_l(x) = (g_0(x), ..., g_l(x))'$ denotes the vector containing the first $l+1$ functions $g_0, ..., g_l$. We are interested in the maximization problem

$$\sup\left\{\prod_{l=0}^{n} (c_l' A_l^{-1}(\xi)\, c_l)^{-\beta_l} \,|\, \xi \in \Xi\right\}, \tag{R*}$$

where $\beta_0, ..., \beta_n$ are positive numbers with sum 1 and Ξ is the set of all probability measures defined on the interval $[-1, 1]$ such that $A_0(\xi), ..., A_n(\xi)$ are nonsingular. Note that for the choices

$$g_l(x) = x^l \tag{A.2}$$

and

$$g_l(x) = \sqrt{1-x^2}\, x^l \tag{A.3}$$

(A.1) gives the matrices $\underline{M}_{2l}(\xi)$ and $\bar{M}_{2l+2}(\xi)$ defined by (2.1) and (2.2), respectively. From the identity $c_l' A_l^{-1}(\xi)\, c_l = \det(A_{l-1}(\xi))/\det(A_l(\xi))$ it is obvious that for (A.2) and (A.3) the problem (R*) is equivalent to the problem (P*) and (Q*) (note that in the case (A.2) we define the product over the indices $l = 1, ..., n$ and that we have written (R*) as a maximization problem). For simplification of the notation we define $j_l(A_l) := (c_l' A_l^{-1} c_l)^{-1} = \det(A_l)/\det(A_{l-1})$ ($A_l \in \mathbb{R}^{(l+1)\times(l+1)}$ positive definite) and $j(a_0, ..., a_n) := \prod_{l=0}^{n} a_l^{\beta_l}$ for positive numbers $a_0, ..., a_n$. The function appearing in the problem (R*) can then be written as $j(j_0(A_0(\xi)), ..., j_n(A_n(\xi)))$.

LEMMA A.1. *The function* $j(a_0, ..., a_n) = \prod_{l=0}^{n} a_l^{\beta_l}$ *is superadditive and concave on* $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$ *(here* $\mathbb{R}_+$ *denotes the set of positive real numbers).*

Proof. Obviously, concavity is an immediate consequence of superadditivity. For the proof of this property we use the well known inequality (see [13, p. 190])

$$\prod_{l=1}^{m} x_l^{\alpha_l} \leqslant \sum_{l=1}^{m} \alpha_l x_l \qquad \forall x_l \geqslant 0,\ \alpha_l \geqslant 0,\ \sum_{l=1}^{m} \alpha_l = 1 \tag{A.4}$$

and obtain for positive numbers $a_0, ..., a_n, b_0, ..., b_n$

$$\begin{aligned} j(a_0 + b_0, ..., a_n + b_n) &= \prod_{l=0}^{n} (a_l + b_l)^{\beta_l} \\ &= \left[\prod_{l=0}^{n} a_l^{\beta_l} + \prod_{l=0}^{n} b_l^{\beta_l}\right] \\ &\quad \times \left[\prod_{l=0}^{n} \left(\frac{a_l}{a_l + b_l}\right)^{\beta_l} + \prod_{l=0}^{n} \left(\frac{b_l}{a_l + b_l}\right)^{\beta_l}\right]^{-1} \\ &\geqslant \left[\prod_{l=0}^{n} a_l^{\beta_l} + \prod_{l=0}^{n} b_l^{\beta_l}\right] \\ &\quad \times \left[\sum_{l=0}^{n} \beta_l \frac{a_l}{a_l + b_l} + \sum_{l=0}^{n} \beta_l \frac{b_l}{a_l + b_l}\right]^{-1} \\ &= \prod_{l=0}^{n} a_l^{\beta_l} + \prod_{l=0}^{n} b_l^{\beta_l} = j(a_0, ..., a_n) + j(b_0, ..., b_n). \quad \blacksquare \end{aligned}$$

In what follows let $NND(k)$ and $PD(k)$ denote the set of all nonnegative and positive definite $k \times k$ matrices. Defining

$$\mathcal{N} := \left\{(B_0, ..., B_n) \mid B_l \in NND(l+1),\ \sum_{l=0}^{n} \beta_l f_l'(x)\, B_l f_l(x) \leqslant 1\ \forall x \in [-1, 1]\right\}$$

we have the following result (note that the next and the following theorem generalize the corresponding results in [11] where the special case $n = 0$ and a more general "information" function $j_0(\cdot)$ is considered).

THEOREM A.2. *For every $\xi \in \Xi$ and for every $N = (N_0, ..., N_n) \in \mathcal{N}$ we have*

$$j(j_0(A_0(\xi)), ..., j_n(A_n(\xi))) \leqslant \prod_{l=0}^{n} (c_l' N_l c_l)^{-\beta_l} \tag{A.5}$$

with equality if and only if

$$N_l = \frac{A_l^{-1}(\xi)\, c_l c_l' A_l^{-1}(\xi)}{c_l' A_l^{-1}(\xi)\, c_l}, \qquad l = 0, ..., n.$$

Proof. The proof given in [11] for the special case $n = 0$ can be transferred to the situation considered in this paper. For completeness the main steps are given here (see also [4] for more details). From the proof of Theorem 3 in [11] we obtain

$$\operatorname{tr}(A_l'(\xi)\, N_l) \geqslant \frac{c_l' N_l c_l}{c_l' A_l^{-1}(\xi)\, c_l} \tag{A.6}$$

with equality if and only if

$$A_l^{1/2}(\xi)\, N_l^{1/2} = \frac{A_l^{-1/2}(\xi)\, c_l c_l' N_l^{1/2}}{c_l' A_l^{-1}(\xi)\, c_l}, \qquad l = 0, ..., n. \tag{A.7}$$

This implies for all $N = (N_0, ..., N_n) \in \mathcal{N}$ and $\xi \in \Xi$ (integrating the restriction in the definition of the set $\mathcal{N}$ with respect to the measure $d\xi(x)$)

$$\begin{aligned} 1 &\geqslant \sum_{l=0}^{n} \beta_l \int_{-1}^{1} f_l'(x)\, N_l f_l(x)\, d\xi(x) \\ &= \sum_{l=0}^{n} \beta_l \operatorname{tr}(A_l'(\xi)\, N_l) \\ &\geqslant \sum_{l=0}^{n} \beta_l \frac{c_l' N_l c_l}{c_l' A_l^{-1}(\xi)\, c_l} \\ &\geqslant \prod_{l=0}^{n} (c_l' N_l c_l)^{\beta_l} \cdot j(j_0(A_0(\xi)), ..., j_n(A_n(\xi))) \end{aligned} \tag{A.8}$$

and proves (A.5). Here we have used the inequalities (A.4) and (A.6) in the last two steps. Equality in (A.5) implies $c_l' N_l c_l = c_l' A_l^{-1}(\xi)\, c_l$ (this follows from the equality in (A.4) and (A.8)) and a postmultiplication of (A.7) by $N_l^{1/2} c_l$ yields $A_l^{-1}(\xi)\, c_l = N_l c_l$ $(l = 0, ..., n)$. Thus we obtain from (A.7)

$$N_l = \frac{A_l^{-1}(\xi)\, c_l c_l' A_l^{-1}(\xi)}{c_l' A_l^{-1}(\xi)\, c_l}, \qquad l = 0, ..., n$$

which completes the proof of Theorem A.2. ∎

THEOREM A.3.

$$\sup\{j(j_0(A_0(\xi)), ..., j_n(A_n(\xi)))\,|\,\xi \in \Xi\}$$
$$= \min\left\{\prod_{l=0}^{n} (c_l' N_l c_l)^{-\beta_l}\,|\,(N_0, ..., N_n) \in \mathcal{N}\right\}. \tag{A.9}$$

Proof. The proof follows essentially the steps of the proof (for the case $n=0$ and a more general function $j_0(\cdot)$) given in [11]. Let $\mathcal{M}$ denote the set of all "matrices"

$$(A_0(\xi), ..., A_n(\xi)) \in \mathbb{R}^{1\times 1} \times \cdots \times \mathbb{R}^{(n+1)\times(n+1)}$$

defined by (A.1) ($\xi \in \Xi$) and define the functions

$$f(A) = \begin{cases} 0 & \text{if } \ A = (A_0, ..., A_n) \in \mathcal{M} \\ \infty & \text{else } (A \in \mathbb{R}^{1\times 1} \times \cdots \times \mathbb{R}^{(n+1)\times(n+1)}) \end{cases}$$

$$g(A) = \begin{cases} \log j(j_0(A_0), ..., j_n(A_n)) \\ \qquad \text{if } \ A = (A_0, ..., A_m),\ A_l \in PD(l+1) \\ -\infty \\ \qquad \text{else } (A \in \mathbb{R}^{1\times 1} \times \cdots \times \mathbb{R}^{(n+1)\times(n+1)}). \end{cases}$$

It is straightforward to show that the mappings $A_l \to j_l(A_l) = (c_l' A_l^{-1} c_l)^{-1}$ are concave (on $PD(l+1)$) and it follows from Lemma A.1 that the function g is concave on $\mathcal{M}$. Fenchel's Duality Theorem (see, e.g., [14, p. 327]) provides a general duality result for the difference of a concave function g and a convex function f and the difference of the corresponding conjugate functions (note that the function f is only used for the definition of the set $\mathcal{M}$). Here we need a slight modification of this result using the "weighted" inner product

$$\langle A, B\rangle := \sum_{l=0}^{n} \beta_l \operatorname{tr}(A_l' B_l)$$

($A = (A_0, ..., A_n)$, $B = (B_0, ..., B_n) \in \mathbb{R}^{1\times 1} \times \cdots \times \mathbb{R}^{(n+1)\times(n+1)}$) instead of the common inner product. Thus we obtain from Fenchel's Duality Theorem

$$\sup_A \{g(A) - f(A)\} = \min_B \{f^*(B) - g^*(B)\}, \tag{A.10}$$

where f^* and g^* are the corresponding conjugate functions of f and g (see [13, p. 30] or [14]) defined by

$$g^*(B) = \inf\left\{\sum_{l=0}^{n} \beta_l \operatorname{tr}(A_l' B_l) - g(A) \mid A_l \in PD(l+1),\ l = 0, ..., n\right\}$$

$$f^*(B) = \sup\left\{\sum_{l=0}^{n} \beta_l \operatorname{tr}(A_l' B_l) \mid (A_0, ..., A_n) \in \mathcal{M}\right\}.$$

The functions f^* and g^* have the same value at $B = (B_0, ..., B_n)$ and $\frac{1}{2}[B + B']$ and the minimization problem can be carried out over the set of "vectors" whose components are symmetric matrices $B_l \in \mathbb{R}^{(l+1)\times(l+1)}$. For such "vectors" B we have

$$\begin{aligned} g^*(B) &= \inf\left\{\sum_{l=0}^{n} \beta_l [\operatorname{tr}(A_l' B_l) - \log j_l(A_l)] \mid A_l \in PD(l+1),\ l = 0, ..., n\right\} \\ &= \sum_{l=0}^{n} \beta_l \inf\{\operatorname{tr}(A_l' B_l) - \log j_l(A_l) \mid A_l \in PD(l+1)\} \\ &= 1 + \sum_{l=0}^{n} \beta_l \log(c_l' B_l c_l), \end{aligned}$$

where we have used the identity $(l = 0, ..., m)$

$$\inf\{\operatorname{tr}(A_l' B_l) - \log j_l(A_l) \mid A_l \in PD(l+1)\} = 1 + \log(c_l' B_l c_l)$$

in the last step, (which was proved by Pukelsheim in [11, p. 346]). For every $B = (B_0, ..., B_n) \neq 0$ (with symmetric components $B_l \in \mathbb{R}^{(l+1)\times(l+1)}$ and $g^*(B) > -\infty$) we obtain from the definition of f^* that $f^*(B)$ is positive and that the function

$$h(\alpha) = f^*(\alpha B) - g^*(\alpha B) = \alpha f^*(B) - 1 - \log \alpha - \sum_{l=0}^{n} \beta_l \log(c_l' B_l c_l)$$

attains it unique minimum at $\alpha = 1/f^*(B)$. This minimum is given by

$$\log f^*(B) - \sum_{l=0}^{n} \beta_l \log(c_l' B_l c_l) = \sum_{l=0}^{n} \log(c_l' N_l c_l)^{-\beta_l},$$

where $N_l = B_l / f^*(B)$ and the vector $N = (N_0, ..., N_n) \in \mathcal{N}$ by the definition of f^*. The assertion of Theorem A.3 now follows observing that the minimization of the right side of (A.10) can be carried out over the set $\mathcal{N}$ and that the left side corresponds to the left side in (A.9). ▌

Proof of Theorem 2.1. We are proving the assertion for the extremal problem (P); the other case is treated in the same way. Thus we have $\Xi = \underline{\Xi}$, $g_l(x) = x^l$, and $A_l(\xi) = \underline{M}_{2l}(\xi)$ $(l = 0, ..., n)$. We assume that all weights β_l are positive; the case of a vanishing weight β_l is obtained considering the corresponding optimization problems without the lth component. Observing that ($c_l = (0, ..., 0, 1)' \in \mathbb{R}^{l+1}$)

$$c_l' N_l c_l = c_l' c_l c_l' N_l c_l c_l' c_l = c_l' a_l a_l' c_l = (c_l' a_l)^2,$$

where $a_l = (a_0, ..., a_{ll})' \in \mathbb{R}^{l+1}$ is a vector such that $a_l a_l'$ is a full rank decomposition of $c_l c_l' N_l c_l c_l'$ it can easily be shown that the set $\mathcal{N}$ in (A.9) can be replaced by

$$\left\{(a_1 a_1', ..., a_n a_n') \mid a_l \in \mathbb{R}^{l+1}, \sum_{l=1}^{n} \beta_l (a_l' f_l(x))^2 \leqslant 1\right\}.$$

Identifying the vector $a_l \in \mathbb{R}^{l+1}$ as the vector of the coefficients of a polynomial $P_l \in \mathbb{P}_l$ we see that this set coincides with the set $\mathscr{P}_n$ defined in (1.1). Thus the duality in (2.3) is a direct consequence of Theorem A.3 (note that $j_l(A_l(\xi)) = (c_l' \underline{M}_{2l}^{-1}(\xi)\, c_l)^{-1} = \Delta_{2l}(\xi)/\Delta_{2l-2}(\xi)$ and that $c_l' a_l = a_{ll} = m_l(P_l)$ where $P_l(x) = a_l' f_l(x)$). For the proof of the second part let $\underline{\xi}^*$ and $(P_1, ..., P_n)$ denote the optimal solutions of (P*) and (P) ($P_l(x) = a_l' f_l(x)$). Then we have equality in Theorem A.2 for $\underline{\xi}^*$ and $N_l = a_l a_l'$. Thus the second part of this theorem shows

$$a_l = \pm \frac{\underline{M}_{2l}^{-1}(\underline{\xi}^*)\, c_l}{(c_l' \underline{M}_{2l}^{-1}(\underline{\xi}^*)\, c_l)^{1/2}}, \qquad l = 1, ..., n$$

and we obtain for the polynomials P_l

$$P_l(x) = a_l' f_l(x) = \pm \sqrt{\frac{\Delta_{2l}(\underline{\xi}^*)}{\Delta_{2l-2}(\underline{\xi}^*)}}\, c_l' \underline{M}_{2l}^{-1}(\underline{\xi}^*)\, f_l(x), \qquad l = 1, ..., n.$$

Using this representation it follows from

$$\begin{aligned}\int_{-1}^{1} P_l(x)\, f_l'(x)\, d\underline{\xi}^*(x) &= a_l' \int_{-1}^{1} f_l(x)\, f_l'(x)\, d\underline{\xi}^*(x) = a_l' \underline{M}_{2l}(\underline{\xi}^*) \\ &= \pm \frac{c_l'}{(c_l' \underline{M}_{2l}^{-1}(\underline{\xi}^*)\, c_l)^{1/2}} \\ &= \pm \frac{(0, ..., 0, 1)}{(c_l' \underline{M}_{2l}^{-1}(\underline{\xi}^*)\, c_l)^{1/2}}\end{aligned}$$

and

$$\int_{-1}^{1} P_l^2(x)\, d\underline{\xi}^*(x) = \int_{-1}^{1} (a_l' f_l(x))^2\, d\underline{\xi}^*(x) = a_l' \underline{M}_{2l}(\underline{\xi}^*)\, a_l = 1$$

that the polynomials $\{P_l(x)\}_{l=1}^{n}$ are orthonormal with respect to the measure $d\underline{\xi}^*(x)$. ∎

APPENDIX B: PROOF OF LEMMA 3.3

We will show by induction that

$$\begin{aligned}\tilde{P}_{k-1}(x) &:= 2^{k-1}\frac{\Gamma(w/2+a+k)}{\Gamma((w+2)/2+a)\,\Gamma(k)}\,G_{k-1}^{(a,w)}(x)\\ &= \sum_{j=0}^{\lfloor (k-1)/2\rfloor}(-1)^j\frac{\Gamma(a+j)}{\Gamma(a)\,\Gamma(j+1)}\,\frac{\Gamma(w+a+1+j)}{\Gamma(w+a+1)}\\ &\quad\times\frac{\Gamma(k-j)}{\Gamma(k)}\,C_{k-2j-1}^{(w/2+1+a+j)}(x).\end{aligned} \tag{B.1}$$

For $k=1$ the identity (B.1) is obvious while the case $k=2$ yields

$$\tilde{P}_1(x)=2\left(\frac{w+2}{2}+a\right)x=C_1^{((w+2)/2+a)}(x)$$

which is evident from the definition of the ultraspherical polynomials (see Abramowitz and Stegun [1, p. 794]). For the induction step from k to $k+1$ it is convenient to distinguish the cases of odd or even k and we will only consider the case of $k=2m$ even (the case $k=2m+1$ is treated in exactly the same way). From the induction hypotheses we obtain for $\tilde{P}_{2m}(x)$ (by an expansion of the determinant)

$$\begin{aligned}\tilde{P}_{2m}(x) &= \frac{2^{2m}\Gamma((w+2)/2+a+2m)}{\Gamma((w+2)/2+a)\,\Gamma(2m+1)}\Bigg[xG_{2m-1}^{(a,w)}(x)\\ &\quad-\frac{(2m-1+a)(2m+a+w)}{(4m-2+2a+w)(4m+2a+w)}\,G_{2m-2}^{(a,w)}(x)\Bigg]\\ &=\sum_{j=0}^{m-1}(-1)^j\frac{\Gamma(a+j)}{\Gamma(a)\,\Gamma(j+1)}\,\frac{\Gamma(w+a+j+1)}{\Gamma(w+a+1)}\\ &\quad\times\frac{\Gamma(2m-j-1)}{\Gamma(2m+1)}\,\{(2m-j-1)(w+2a+4m)\,xC_{2m-1-2j}^{((w+2)/2+a+j)}(x)\\ &\quad-(2m-1+a)(2m+a+w)\,C_{2m-2-2j}^{((w+2)/2+a+j)}(x)\}.\end{aligned}$$

Now let $\tilde{P}_{2m}^{(f)}(x)$ denote the above sum where the summation in the last expression is only performed over the indices $0, 1, ..., f$ ($f\in\{0, 1, ..., m-1\}$, i.e., $\tilde{P}_{2m}^{(m-1)}(x)=\tilde{P}_{2m}(x)$) and similarly define $F_{2m}^{(f)}(x)$ as the "truncated" sum of the polynomial defined on the right side of Eq. (B.1), that is,

$$F_{2m}^{(f)}(x) = \sum_{j=0}^{f} (-1)^j \frac{\Gamma(a+j)}{\Gamma(a)\,\Gamma(j+1)} \frac{\Gamma(w+a+j+1)}{\Gamma(w+a+1)} \times \frac{\Gamma(2m+1-j)}{\Gamma(2m+1)} C_{2m-2j}^{((w+2)/2+a+j)}(x)$$

($f \in \{0, ..., m\}$). Then we have the following lemma whose proof is given in a paper of Dette [5].

LEMMA B.1. *The polynomials* $F_{2m}^{(f)}(x)$ *and* $\tilde{P}_{2m}^{(f)}(x)$ *satisfy the equation* ($f = 0, ..., m-1$)

$$\tilde{P}_{2m}^{(f)}(x) - \tilde{F}_{2m}^{(f)}(x) = (-1)^{f+1} \frac{\Gamma(a+f+1)}{\Gamma(a)\,\Gamma(f+1)} \frac{\Gamma(w+a+2+f)}{\Gamma(w+a+1)} \times \frac{\Gamma(2m-1-f)}{\Gamma(2m+1)} \cdot C_{2m-2f-2}^{((w+2)/2+f+a)}(x).$$

Using Lemma B.1 we obtain now for the differences of the polynomials appearing in (B.1)

$$\begin{aligned}
&\tilde{P}_{2m}(x) - 2^{2m} \frac{\Gamma(w/2+a+2m+1)}{\Gamma((w+2)/2+a)\,\Gamma(2m+1)} G_{2m}^{(a,w)}(x) \\
&= \tilde{P}_{2m}^{(m-1)}(x) - F_{2m}^{(m-1)}(x) - (-1)^m \frac{\Gamma(a+m)}{\Gamma(a)\,\Gamma(m+1)} \\
&\quad \times \frac{\Gamma(w+a+m+1)}{\Gamma(w+a+1)} \frac{\Gamma(m+1)}{\Gamma(2m+1)} = 0
\end{aligned}$$

which establishes (B.1) (in the case of even $k = 2m$). Because the odd case is proved in the same way the assertion of Lemma 3.3 follows.

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